

SOME RESULTS ON THE INTEGRABILITY OF THE CENTER BUNDLE FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

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ABSTRACT. We prove, for f a partially hyperbolic diffeomorphism with center dimension one, two results about the integrability of its central bundle. On one side, we show that, if $\Omega(f) = M$ and $\dim(M) = 3$, the absence of periodic points implies its unique integrability. On the opposite side, we prove that any periodic point $p \in \text{Per}(f)$ of large enough period N has an f^N -invariant center manifold (everywhere tangent to the center bundle).

We also obtain, as a consequence of the last result, that there is an open and dense subset of C^1 robustly transitive and partially hyperbolic diffeomorphisms with center dimension 1, such that either the strong stable or the strong unstable foliation is minimal. This generalizes a result obtained in [BDU] for three-dimensional manifolds to any dimension.

1. INTRODUCTION

In this paper we shall consider partially hyperbolic diffeomorphisms with one dimensional center direction E^c . By a partially hyperbolic diffeomorphism we mean $f \in \text{Diff}(M)$, M a closed manifold, admitting a non trivial Df -invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E_x^\sigma$ with $\sigma = s, c, u$ and $x \in M$ verify:

$$\|Df(x)v^s\| < \|Df(x)v^c\| < \|Df(x)v^u\|$$

for some suitable Riemannian metric, which we call *adapted*. It is also required that the norm of the operators $Df(x)|_{E^s}$ and $Df^{-1}(x)|_{E^u}$ be strictly less than 1. We shall denote $\mathcal{PH}^r(M)$ the family of C^r partially hyperbolic diffeomorphisms of M . Along this paper we will consider only the case $\dim E^c = 1$ and we denote the set of such diffeomorphisms by $\mathcal{PH}_1^r(M)$.

On one hand, it is well known by classical invariant manifold theory that the bundles E^s and E^u are uniquely integrable thus obtaining two foliations called the strong stable and the strong unstable foliations. On the other hand, it is not known in general whether either the center bundle, the center stable ($E^{cs} = E^s \oplus E^c$) or the center unstable ($E^{cu} = E^u \oplus E^c$) are integrable. The

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hypothesis of integrability of this bundles has played an important role in partial hyperbolicity theory, see for instance [HHU2]. Although recent work shows that the integrability assumption can be bypassed to obtain ergodicity (see [BW], [HHU1]) it seems that it remains to play a crucial role if one looks for a topological description or even classification of partially hyperbolic diffeomorphisms (see for instance [BBI]).

In this paper we prove two results about the integrability of the center bundle. In our first theorem we prove that, if $\dim(M) = 3$ and the nonwandering set is the whole manifold, the absence of periodic points implies its unique integrability and, in the second, that periodic points of period N high enough have central curves (tangent at *every* point to E^c) invariant by f^N .

THEOREM 1. *Let $f \in \mathcal{PH}_1^r(M)$ be such that $\text{Per}(f) = \emptyset$ and $\Omega(f) = M$ and assume that $\dim(M) = 3$. Then, E^c is uniquely integrable.*

THEOREM 2. *Let $f \in \mathcal{PH}_1^r$. There exists $K > 0$ such that for any $p \in \text{Per}(f)$ with period $N > K$ there exists, through p , an f^N invariant curve tangent to E^c at every point.*

A C^r , $r \geq 1$, robustly transitive diffeomorphism is a diffeomorphism having a neighborhood in $\text{Diff}^r(M)$ such that every g in this neighborhood is transitive. We show as a consequence of Theorem 2 a generalization of a result in [BDU] to M of any dimension. In the cited paper the same result is proved for any $f \in \mathcal{PH}_1^r(M)$ and $\dim(M) = 3$ or for M of any dimension but assuming unique integrability of the center bundle.

THEOREM 3. *Let $\mathcal{T}(M)$ be the set of C^1 robustly transitive diffeomorphisms. Then, there exists an open and dense subset of $\mathcal{T}(M) \cap \mathcal{PH}_1^r(M)$ such that either the strong stable or the strong unstable foliation is minimal.*

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2. PRELIMINARIES

It is a known fact that, for $f \in \mathcal{PH}^r(M)$, there are foliations \mathcal{W}^σ tangent to the distributions E^σ for $\sigma = s, u$ (see for instance [BP]).

Due to Peano's Theorem, for each $x \in M$ there are curves $\alpha_x(t)$ such that $\alpha_x(0) = x$ and $\dot{\alpha}_x(t) \in E^c(\alpha_x(t)) \setminus \{0\}$ for some open interval of parameters t containing 0. We shall call these curves *central curves through* x , and denote by $W_{loc}^c(x)$ the component of a central curve through x intersected by a small ball. It is easy to see that f takes central curves into central curves.

Denoting the leaf of \mathcal{W}^σ through x by $W^\sigma(x)$, with $\sigma = s, u$, we write, as usual, $W_{loc}^\sigma(x)$ for the connected component of $W^\sigma(x) \cap B(x)$, where $B(x)$ is a small ball

around x . Observe that for any choice of $W_{loc}^c(x)$, the sets

$$W_{loc}^\sigma(W_{loc}^c(x)) = \bigcup_{y \in W_{loc}^c(x)} W_{loc}^\sigma(y) \quad \sigma = s, u$$

are C^1 (local) manifolds tangent to the bundle $E^{c\sigma} = E^\sigma \oplus E^c$ (with $\sigma = s, u$) at every point (see, for instance [BBI]). For further use we will call, respectively, $W_{loc}^{cs}(x)$ and $W_{loc}^{cu}(x)$ the sets obtained as above depending, as it is obvious, on the choice of $W_{loc}^c(x)$.

REMARK 2.1. *Moreover, given $x, y \in M$, for all $W_{loc}^{cs}(x)$ such that $y \in W_{loc}^{cs}(x)$, there exists a central curve $W_{loc}^c(y)$ through y contained in $W_{loc}^{cs}(x)$ (see [BBI])*

Let us say that a set Γ is σ -saturated if Γ is union of leaves of \mathcal{W}^σ , $\sigma = s, u$ and let us call the accessibility class of x , $AC(x)$, the minimal s - and u -saturated set that contains the point x (that is, the set of points that can be joined to x by a us -path). If f has only one accessibility class we say that it satisfies the *accessibility property*.

3. ABSENCE OF PERIODIC POINTS

LEMMA 3.1. *Let $f \in \mathcal{PH}_1^r(M)$ be such that $\mathcal{P}er(f) = \emptyset$ and $\Omega(f) = M$. Then, either f has the accessibility property or E^s and E^u are jointly integrable.*

Proof. Let $\Gamma(f)$ be the set of points such that its accessibility class is not open and suppose that $\emptyset \neq \Gamma(f) \neq M$. Thus, Lemma A.5.1 of [HHU1] implies the existence of a periodic point in $\Gamma(f)$ contradicting that $\mathcal{P}er(f) = \emptyset$.

$\Gamma(f) = M$ is equivalent to the joint integrability of E^s and E^u (see [HHU1]) \square

REMARK 3.2. *Observe that the same proof gives that, for $f \in \mathcal{PH}_1^r(M)$ such that $\mathcal{P}er(f) = \emptyset$ and $\Omega(f) = M$, every closed invariant su -saturated set is either empty or the whole M .*

The following lemma generalizes (with essentially the same proof) Brin's result ([B1]) stating that accessibility implies transitivity. For the sake of completeness we include the proof here.

LEMMA 3.3. *Let $f \in \mathcal{PH}^r(M)$ be such that $\Omega(f) = M$ and assume that every closed invariant su -saturated set is either empty or the whole M . Then f is transitive.*

Proof. Let U and V be two open sets. For all $x \in M$ the set $K = \overline{\bigcap_{i=0}^{\infty} \bigcup_{n \geq i} AC(f^n(x))}$ is invariant, closed, su -saturated and nonempty (observe that $AC(f^n(x)) = f^n(AC(x))$ and so, $K = M$). Then, by taking $x \in U$, we can choose $N \in \mathbb{N}$ such that $AC(f^N(x)) \cap V \neq \emptyset$. Call $U_N = f^N(U)$.

We shall show that there exists $n \in \mathbb{N}$ such that $f^n(U_N) \cap V \neq \emptyset$ which implies $f^{N+n}(U) \cap V \neq \emptyset$. Since U and V are arbitrary open sets the transitivity of f follows from this last property.

The considerations above imply that there is an su -path $[z_0, \dots, z_k]$ with $z_0 \in U_N$ and $z_k \in V$. By continuity of the strong stable and unstable foliations we can choose V_0, V_1, \dots, V_k open sets such that:

- $z_i \in V_i \forall i = 0, \dots, k$.
- $V_0 \subset U_N$ and $V_k \subset V$
- for each point of $x \in V_i$ there exists a su -path $[x = x_i, x_{i+1}, \dots, x_k]$ joining x with a point of V_k with $x_j \in V_j \forall j = i, \dots, k$.

Suppose that the path $[z_0, z_1]$ is tangent to the stable bundle (the unstable case is a little bit easier), then there exists a neighborhood $B \subset V_1$ of z_1 such that each point in it can be joined with V_0 by an s -path of uniformly bounded length (in fact its length can be chosen approximately of the length of $[z_0, z_1]$) and we can chose B in such a way that there exists $\rho > 0$ such that $W^s(x) \cap V_1 \supset W_\rho^s(x) \forall x \in B$. Since $\Omega(f) = M$, there exists an arbitrarily large $m \in \mathbb{N}$ and a point $w \in B$ such that $f^{-m}(w) \in B$. This implies that, if m is large enough, $f^{-m}(W^s(w))$ contains the path joining $f^{-m}(w)$ and V_0 . Thus $f^{-m}(V_1) \cap U_N \neq \emptyset$ which implies $f^m(U_N) \cap V_1 \neq \emptyset$.

Now substitute V_1 by $V_1 \cap f^m(U_N)$ and repeat the procedure. By induction we obtain that there is $n \in \mathbb{N}$ such that $f^n(U_N) \cap V \neq \emptyset$.

□

REMARK 3.4. *It is not the issue of this work to achieve the minimal hypothesis to obtain transitivity by using Brin's argument. However, let us mention that almost the same proof works if one substitutes the hypothesis on the density of every invariant saturated nonempty set by the weaker one demanding the existence of an accessibility class whose orbit by f is dense (there exists x such that $\cup\{f^n(AC(x)); n \in \mathbb{Z}\} = M$).*

The following theorem is a direct corollary of Remark 3.2 and Lemma 3.3.

THEOREM 3.5. *Let $f \in \mathcal{PH}_1^r(M)$ be such that $\text{Per}(f) = \emptyset$ and $\Omega(f) = M$. Then, f is transitive.*

Before proving Theorem 1 let us state the following lemma, which is a consequence of continuity and transversality of the invariant bundles:

LEMMA 3.6. *For $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ and $z \in W_\delta^c(x)$, then $W_{loc}^c(y) \cap W_\varepsilon^s(W_\varepsilon^u(z)) \neq \emptyset$, regardless of the choice of center leaves for x and y .*

In particular, if $W_{loc}^c(y) \subset W_{loc}^{cu}(x)$ then $W_{loc}^c(y) \cap \mathcal{W}_\varepsilon^u(z) \neq \emptyset$ for all $z \in W_\delta^c(x)$

REMARK 3.7. *As a corollary of lemma above, if E^c , restricted to some $W_{loc}^{cu}(x)$, is non uniquely integrable at x , then for sufficiently small $\delta > 0$, and for each connected central subsegment containing x , say c , in one of the two separatrix, there is $N > 0$ for which $f^n(c) \notin B_\delta(f^n(x))$ for all $n \geq N$.*

Proof of Remark 3.7. Take c_1 and c_2 two different center curves, contained in the same component of $W_{loc}^{cu}(x) \setminus W_{loc}^u(x)$ and having x as endpoint. Since c_1 and c_2 are different, there exist $y_1 \neq y_2$ such that $y_2 \in W_{loc}^u(y_1)$ and $y_i \in c_i$, $i = 1, 2$. The exponential growth of $W^u(y_1)$ under the action of f implies that there exists $N > 0$ such that $f^n(y_2) \notin W_\varepsilon^u(f^n(y_1))$ for all $n \geq N$ and, by lemma above, we obtain that $f^n(c_2) \not\subset B_\delta(f^n(x))$. \square

Observe that one can prove without using Lemma 3.6 that either c_1 or c_2 should grow but, in fact, what is proved in Remark 3.7 is that both center curves grow.

Proof of Theorem 1. By Theorem 3.5 we know that f is transitive.

As unique integrability is a local property we can suppose, by taking a double covering and f^2 if necessary, that E^c is oriented and its orientation is preserved by f .

Suppose that E^c is not uniquely integrable at x . Then, there are two different arcs α and β tangent to E^c beginning at x with the same (positive) orientation. By taking intersections of $W_{loc}^{cs}(\alpha)$ and $W_{loc}^{cs}(\beta)$ with some $W_{loc}^{cu}(x)$ we can assume that both arcs are contained in the same $W_{loc}^{cu}(x)$ (in case $W_{loc}^{cs}(\alpha) \subset W_{loc}^{cs}(\beta)$ we can do the same argument for f^{-1})

Since $W_{loc}^{cu}(x)$ is two dimensional α , β and a conveniently chosen unstable arc bound an open region U of $W_{loc}^{cu}(x)$ and, for every point z in U , there is a center arc γ inside $W_{loc}^{cu}(x)$ joining (in the positive orientation) x with z . As f is transitive, taking the intersection of the strong stable manifold of a point with dense forward orbit with U , we can choose z such that its forward orbit is dense and γ with length much less than the δ of Remark 3.7.

Now Remark 3.7 implies that for $n > N$ the length of $f^n(\gamma)$ is larger than δ . We can take from z a center continuation of γ in the positive direction and a point w in it, close to z , but not in γ .

Consider $C = W_\varepsilon^u(\gamma)$ for ε small.

Then there is K very large (in particular larger than N) such that $f^K(z)$ is very close to w . Since the length of $f^K(\gamma)$ is larger than δ and the unstable manifolds growth exponentially, the projection of $f^K(C)$ to $W_{loc}^{cu}(x)$ contains C . This implies that there is a f^K -invariant strong stable manifold and thus, we obtain a periodic point.

\square

REMARK 3.8. *In fact with the same argument can be proved that if $f \in \mathcal{PH}_1^r(M)$ satisfies $\dim(E^s) = 1$, $\Omega(f) = M$ and $\mathcal{P}er(f) = \emptyset$, E^{cu} is uniquely integrable.*

If E^s and E^u are jointly integrable the assumptions on the dimension of the the strong bundles and the nonwandering set are not needed to obtain the unique integrability of the one dimensional center bundle E^c .

In order to prove next theorem we need the following standard lemma:

LEMMA 3.9. *There is $\varepsilon_0 > 0$ such that if $x \in \Gamma(f)$ verifies $f^k(B_{\varepsilon_0}^{su}(x)) \cap B_{\varepsilon_0}^{su}(x) \neq \emptyset$ for some $k > 0$, then there is a periodic point in $B_{\varepsilon_0}^{su}(x)$.*

THEOREM 3.10. *Let $f \in \mathcal{PH}_1^r(M)$ be such that $\text{Per}(f) = \emptyset$ and E^s and E^u are jointly integrable. Then E^c is uniquely integrable.*

Proof. If E^c is not uniquely integrable at $x \in M$ then there exist two central curves α and β through x . As in the proof of Theorem 1, by possibly taking intersections, we may assume, for instance that $\beta \subset W_{loc}^u(\alpha)$.

Consider three different points $w_2 < w_1 < w_3$ in $\omega(x)$, such that $d(w_i, w_j) < \delta/4$, $i, j = 1, 2, 3$. This is possible since f has no periodic points, and $E^s \oplus E^u$ is a codimension one bundle, so we can suppose that w_1 is locally between $W^{su}(w_2)$ and $W^{su}(w_3)$, integral manifolds of $E^s \oplus E^u$. Take n_1 such that $d(f^{n_1}(x), w_1) < \delta/16$ (δ as in Remark 3.7) We are assuming that E^c is oriented and that f preserves its orientation (modulo taking a double covering and f^2 , if necessary). Take a small arc γ in $f^{n_1}(\alpha)$ beginning at $f^{n_1}(x)$. As E^c is not uniquely integrable at $f^{n_1}(x)$ the previous observation implies that, we may take n_2 large enough so that $f^{n_2}(x)$ be $\delta/16$ -near w_2 , and the length of $f^{n_2-n_1}(\gamma)$ be greater than δ . Now, projecting locally via the su -foliation we obtain a map of the interval and as a consequence there is a point e in an su -leaf such that $f^{n_2-n_1}(e)$ is in the same su -leaf. This can be made so that Lemma 3.9 applies, whence we would have a periodic point.

This implies the unique integrability at $f^{n_1}(x)$ and, of course, at x on one direction of E^c . To obtain the unique integrability on the other direction we argue in the same way with w_3 instead of w_2 . \square

We remark that in Theorems 1 and 3.10 we prove not only the existence of a foliation tangent to E^c but its uniqueness.

4. EXISTENCE OF CENTRAL CURVES FOR PERIODIC POINTS

This section is devoted to prove Theorem 2.

Proof of Theorem 2. By the standard Center Manifold Theorem, through any periodic point p there exists an immersed, invariant by the period N of p , curve γ such that it is tangent at p to $E^c(p)$ and it is invariant when it make sense. After that, take a connected component of $\gamma \setminus p$ say γ_1 . Suppose for a while that the center eigenvalue at p is positive. Then, we have to situations: either $f^N(\gamma_1) \subset \gamma_1$ or $f^{-N}(\gamma_1) \subset \gamma_1$ (the simultaneous occurrence of both situations is possible).

Suppose that we are in the first case. Then, as in [BBI], $W_{loc}^s(\gamma_1) = \cup_{x \in \gamma_1} W_{loc}^s(x)$ is a C^1 f^N -invariant manifold tangent to E^{cs} at every point. Analogously, for the second case and $W_{loc}^u(\gamma_1)$. As a conclusion we have that associated to each component of γ we obtain either a center stable or a center unstable manifold invariant by f^N or f^{-N} respectively. In case the center eigenvalue were negative we can do the same procedure for f^{2N} obtaining that either $W_{loc}^s(\gamma)$ or $W_{loc}^u(\gamma)$ is respectively f^N or f^{-N} -invariant. In fact, if for example $W_{loc}^s(\gamma_1)$ is invariant by f^{2N} then $W_{loc}^s(\gamma_1) \cup f^N(W_{loc}^s(\gamma_1))$ is invariant by f^N .

Suppose that, without loss of generality, $W_{loc}^s(\gamma_1)$ is f^N -invariant.

We shall make use of the following property: given $\varepsilon > 0$ there exist $\rho > 0$ and $\tau > 0$ such that if $y \in W_\rho^c(x)$ for some center manifold of x , any center curve beginning at $W_\tau^s(y)$ and contained in $W^{cs}(x)$ intersects (with length near ρ) $W_{\varepsilon/2}^s(x)$. Moreover, given $\alpha > 0$ there exists $\varepsilon > 0$ such that if $x \in W_\alpha^{cs}(y)$ then $W_\varepsilon^s(x)$ intersects any center manifold $W^c(y)$ contained in $W^{cs}(y)$.

Fix a small $\alpha > 0$ and take ε with the preceding property and $\rho < \alpha$ and τ as above. Finally, take K such that $\lambda^K \varepsilon < \tau/3$, where λ is such that the norm of $Df(x)|_{E^s}$ is less than $\lambda < 1$.

Now suppose that $N > K$ and take $x \in W_r^c(p)$, $x \neq p$, and $r < \alpha$ so small that the length of $f^N(W_r^c(p))$ is smaller than ρ .

Then,

- $W_\varepsilon^s(x)$ intersects any center curve beginning at p .
- $f^N(W_\varepsilon^s(x))$ cuts a center manifold that contains x in a point y
- $f^N(W_\varepsilon^s(x)) \subset W_\tau^s(y)$

Let $\{X_n\}$ be a sequence of C^1 line fields defined in $W_{loc}^{cs}(p)$ and converging in the C^0 topology to E^c . Observe that, for n big enough, the integral curves of X_n intersecting $W_\tau^s(y)$ also cut $W_\varepsilon^s(x)$. Then, consider the following maps $\varphi_n : W_\varepsilon^s(x) \rightarrow W_\varepsilon^s(x)$. First, for $z \in W_\varepsilon^s(x)$, take $f^N(z)$ and, after this, since $f^N(z) \in W_\tau^s(y)$, take the point of intersection of the solution of X_n through $f^N(z)$ with $W_\varepsilon^s(x)$. By Brower's Theorem φ_n has a fixed point. This means that there exists $w_n \in W_\varepsilon^s(x)$ such that $f^N(w_n)$ and w_n are in the same integral curve for X_n . Arzela-Ascoli's Lemma implies that we have a limit center curve and a point w in it such that $f^N(w) \in W_{loc}^c(w)$. If $f^{kN}(w) \rightarrow_{k \rightarrow \infty} p$ we obtain the invariant center curve through p by iteration of the obtained above. If $f^{kN}(w)$ does not converge to p then, it is easy to prove, that there is another periodic point p_1 in γ_1 and a center arc joining p and p_1 that verifies the theorem. \square

5. MINIMALITY OF STRONG FOLIATIONS

The proof of Theorem 3 is identical to that of the corresponding Theorem of [BDU] by observing that Theorem 2 substitutes Lemma 5.2 of [BDU].

All the discussion in [BDU] about the non orientability of the center bundle is nowadays solved thanks to the open and denseness of the diffeomorphisms with the accessibility property among the ones with one dimensional center bundle (see [DW], [HHU1])

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